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Attractor bifurcation and on-off intermittency

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We analyze the bifurcation of an attractor in a system of multiplicatively coupled maps. It is shown that the phenomenon can be characterized by on-off intermittency and it is rigorously described by the on-off intermittency in the presence of noise through the concept of a virtually invariant manifold. Presenting numerical results which conform our claims, we elucidate the role of on-off intermittency in bifurcation phenomena.

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Intermittency is characterized by temporarily repeating bursting out of invariant objects such as fixed points, periodic orbits, and invariant subspaces, etc., and relatively long evolution in their neighborhood [1–5]. Recently, in connection with the synchronization phenomenon and its various applications, on-off intermittency has attracted wide attention [3]. The bifurcation parameter of on-off intermittency is on a dynamic time-dependent forcing which contrasts with other models such as Pomeau-Manneville intermittency types [2] and crisis-induced intermittency [6] for which parameters are static.

In the synchronized regime the dynamics of a coupled system is restricted to an invariant subspace. The problem of synchronization of coupled systems is thus transmuted as a problem of finding invariant subspace and analyzing its stability in coupled systems with multiplicative driving:

$$x_{n+1}=\Lambda(x_n,y_n)F(x_n), \tag{1}$$

$$y_{n+1}=G(x_n,y_n), \tag{2}$$

where $x_n\in\mathcal{R}^{N_-}, y_n\in\mathcal{R}^{N_+}$ with $\mathcal{R}^{N_-}\cup\mathcal{R}^{N_+}\equiv\mathcal{R}^N$. The invariant manifold is a special solution which is satisfied with $\Lambda(x^*,y_n)F(x^*)=x^*,\ \forall y_n\in\mathcal{R}^{N_+}$. The only generic solution with the nonvanishing interaction ($\Lambda\neq 0$) is $x^*=0$ with $F(0)=0$ since the evolution kernel is vanished in this case only. The solution $x=0$ is fixed point in $\mathcal{R}^{N_-}$ space as well as invariant subspace in $\mathcal{R}^N$ full phase space.

Recently Stefański *et al.* [5] presented a model that exhibits on-off intermittency after the destruction of a torus attractor that is not on the invariant manifold. Moreover, Rim *et al.* [7], showed that on-off intermittency could be a possible route to chaos in random dynamical systems. Their investigations open new aspects of on-off intermittency as well as present the possibility that on-off intermittency plays a role of a new route to chaos in higher dimensional chaotic systems.

In this Rapid Communication, we investigate the bifurcation phenomena of an attractor which appears in coupled systems. We also show that the phenomena can be understood as a process of losing stability of a properly defined virtually invariant manifold. At a bifurcation point the temporal behavior of the system is governed by on-off intermittency in the presence of noise.

Let us consider the two-dimensional coupled system,

$$x_{n+1}=\Lambda(y_n,\lambda,\epsilon)f(x_n), \tag{3}$$

$$y_{n+1}=g(y_n), \tag{4}$$

where we take logistic map $x_n(1-x_n)$ for $f(x_n)$ and Bernoulli map $2y_n \mod 1$ for $g(y_n)$ [1], and $\Lambda(y_n,\lambda,\epsilon)=\lambda+\epsilon(y_n-0.5)$. Here we consider the parameter region $\lambda\in[0,4.0]$ with the coupling strength $\epsilon>0$ acting as noise intensity.

In the bifurcation diagram, one can see two special regions A and B in which the topology of the attractors is changed. The bifurcation points in these regions corresponds to the transcritical and supercritical bifurcation points of logistic map when the coupling strength ϵ is turned off. The bifurcation phenomenon in region A of Fig. 1 has been extensively studied in connection with the synchronization and its practical applications, e.g., secure communication, signal amplification, etc. [8]. Now it is understood as a process of losing stability of invariant manifold and its temporal behavior is characterized by on-off intermittency [3].

So far, the bifurcation phenomenon like in region B is considered to be a kind of crisis phenomenon [6,9], and it has not been extensively studied with the reason that this phenomenon cannot be directly related to a synchronization manifold. According to Ref. [9], a backward tangent bifurcation should cause a tunnel effect between two formerly disjunct attractor bands in the region B, so that an

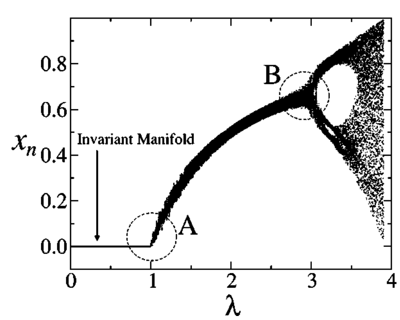

FIG. 1. Bifurcation diagram of Eq. (3) when $\epsilon=0.2$. In region A the invariant subspace $x^*=0$ loses its stability, giving rise to on-off intermittency. How can we understand the route of the bifurcation in region B?

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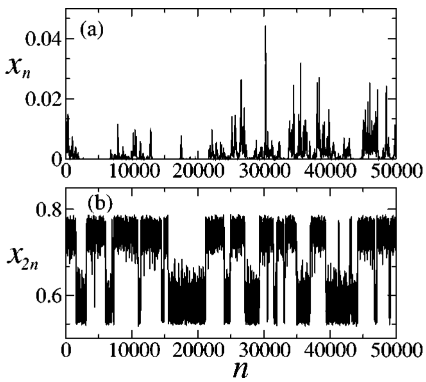

FIG. 2. Time series near the onset of bifurcations. (a) shows typical on-off intermittency at $\lambda=1.003$ and $\epsilon=0.2$ (region *A* in Fig. 1). (b) shows temporal behavior of x_{2n} vs n at $\lambda=3.070$ and $\epsilon=0.2$ (region *B* in Fig. 1).

intermittent-type dynamics should be induced by this sort of crisis, which was then called a tunnel crisis. The characteristic exit times from one attractor band into another was found to scale with the coupling strength ϵ in a different way as compared with the typical crisis-induced intermittency [6].

Instead, here we will analyze the bifurcation phenomenon of the region *B* in terms of on-off intermittency. As we shall see, this phenomenon can be formally described as on-off intermittency in the presence of noise by introducing the virtually invariant manifold.

Figure 2 is showing temporal behaviors for different parameters. We see that the temporal behavior in Fig. 2(a) is a typical on-off intermittency [3]. It is known that the latter is closely related to the loss of the stability of the invariant manifold. However, it seems that the behavior in Fig. 2(b) cannot be classified with any known type of intermittency [2,3] because its bifurcation parameter is dynamical and the phenomenon is independent of the invariant manifold.

Let us first discuss region *A* in Fig. 1. The onset point of the on-off intermittency can be determined by evaluating the transverse Lyapunov exponent. Expanding Eq. (3) around the invariant manifold $x_n=0$ one sees that $x_n=\Lambda(y_n,\lambda,\epsilon)[f(0)+f'(0)(x_n)+O(x_n^2)]$. After τ iterations, one gets $x_{n+\tau}=\Pi_{i=0}^{\tau-1}[\Lambda(y_{n+i},\lambda,\epsilon)f'(0)]x_n$. This allows us to introduce the transverse Lyapunov exponent such that $x_{n+\tau}/x_n=\Pi_{i=0}^{\tau-1}\Lambda(y_{n+i},\lambda,\epsilon)\sim\exp(\lambda_t\tau)$ and finally, we get the following expression:

$$\lambda_t=\frac{1}{\tau}\sum_{i=1}^{\tau-1}\ln[\Lambda(y_{n+i},\lambda,\epsilon)f'(0)]. \tag{5}$$

When the transverse Lyapunov exponent becomes positive the intermittent bursting is started. We can easily derive the transverse Lyapunov exponent since the invariant density of Bernoulli map $g(y_n)$ is uniform so that the exponent is given by $\lambda_t=\int_0^1\ln[\Lambda(y,\lambda,\epsilon)f'(0)]\rho(y)dy=\log(\lambda+\epsilon/2)(1/2+\lambda/\epsilon)+\log(\lambda-\epsilon/2)(1/2-\lambda/\epsilon)-1$. The onset condition $\lambda_t=0$ enables us to obtain the onset parameter value $\lambda=\sqrt{1+\epsilon^2/4}$.

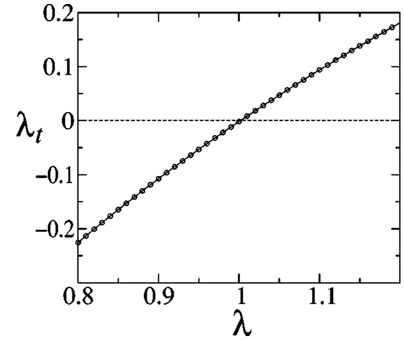

FIG. 3. Transverse Lyapunov near the invariant manifold with $\epsilon=0.2$.

In Fig. 3, one can identify the onset point of on-off intermittency, which is also evaluated from the above equation $\lambda=1.0049\ldots$ for $\epsilon=0.2$. After the onset point, the invariant manifold becomes unstable and the intermittent bursting starts. The difficulty in analyzing the phenomenon of region *B* from Fig. 1 in an analogous way is that it is not a phenomenon near the invariant manifold.

In order to describe the attractor bifurcation phenomenon in Fig. 4, we need to define an appropriate measure that enables us to identify the bifurcation point. The situation is not the same as in the previous case in which we could linearize around the invariant manifold. As one can see in Figs. 4(a)–4(c), after the bifurcation the attractor is separated in the x_n direction. Accordingly, if we define the local Lyapunov exponent in terms of the x_n direction, it would describe the bifurcation phenomenon appropriately. Consider a point on attractor of Fig. 4(a), say x_n, and its nearby point $x_n'=x_n+\delta x_n$. After the one evolution the two points mapped to x_{n+1} and x_{n+1}' such that: $x_{n+1}=\Lambda(y_n,\lambda,\epsilon)f(x_n)$ and $x_{n+1}'=\Lambda(y_n,\lambda,\epsilon)f(x_n')$. After expanding the last equation around x_n and taking τ iteration, we see that $\delta x_{n+\tau}/\delta x_n=\Pi_{i=0}^{\tau-1}\Lambda(y_{n+i},\lambda,\epsilon)f'(x_n)\sim\exp(\lambda_x\tau)$, which enables us to define the Lyapunov exponent in the x_n direction

$$\lambda_x=\frac{1}{\tau}\sum_{i=0}^{\tau-1}\ln(|\Lambda(y_{n+i},\lambda,\epsilon)f'(x_n)|). \tag{6}$$

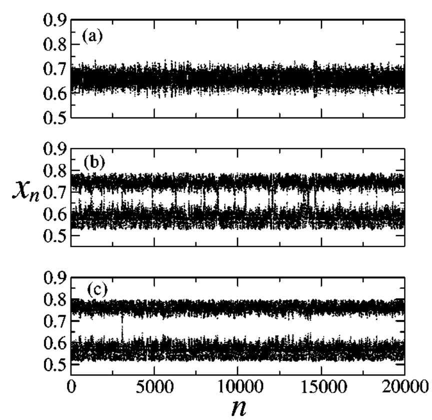

FIG. 4. Time series in region *B*. (a) Before the bifurcation $\lambda=2.95$ (b) onset of bifurcation $\lambda=3.07$, and (c) after the bifurcation $\lambda=3.10$.

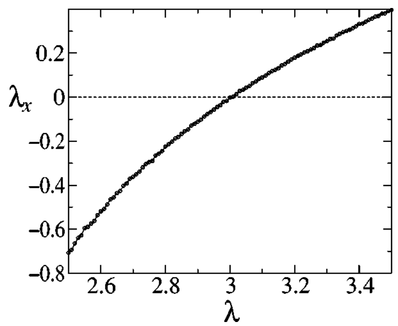

FIG. 5. Transverse Lyapunov exponent of the virtually invariant manifold when $\epsilon = 0.2$.

The above definition describes the growth of the relative distance between two points on an attractor in the x direction, in contrast to the former definition of a transverse Lyapunov exponent, which measures the absolute distance from the invariant manifold. Now, we introduce the virtually invariant manifold $\mathcal{V}$, which is a manifold in $\mathcal{R}^2$ such that: $\mathcal{V} \equiv \{(x^*, y_n) | x^* = \lim_{\epsilon \to 0} \Lambda(y_n, \lambda, \epsilon) f(x^*) \ \ \forall y_n\}$. That is to say, $\mathcal{V}$ is an invariant manifold when the coupling is vanished.

Switching on ϵ means inducing perturbations around the virtually invariant manifold. At every value y_n there exists an instantaneously invariant manifold $\{(x^*, y_n) | x^* = \Lambda(y_n, \lambda, \epsilon) f(x^*)\}$; together they form a cloud around the virtually invariant manifold, so that instead of analyzing merging attractor bands one can analyze the stability of the ''center'' manifold of this cloud, i.e., the stability of the virtually invariant manifold. By combining Eq. (6) with the virtually invariant manifold, we can find the transverse Lyapunov exponent in this regime: $\lambda_x = 1/\tau \Sigma_{i=0}^{\tau-1} \ln[|\Lambda(y_{n+i}, \lambda, \epsilon) f'(x^*)|]$, where $(x^*, y_n) \in \mathcal{V}$. Figure 5 is showing the calculated Lyapunov exponent. By presenting this figure, we want to explain the bifurcation phenomenon in terms of the stability of the virtually invariant manifold. We can see that the virtually invariant manifold in our coupled systems becomes unstable in the x_n direction from $\lambda \approx 3.0$ and the attractor bifurcates.

Figure 6 shows the attractor in the whole phase space. We emphasize that this bifurcation phenomenon is clearly distinguished from the so-called symmetry breaking and increasing bifurcation [4] as well as conventional on-off intermittency [3], because our attractor does not have any discrete symmetry and it is not related to invariant manifold.

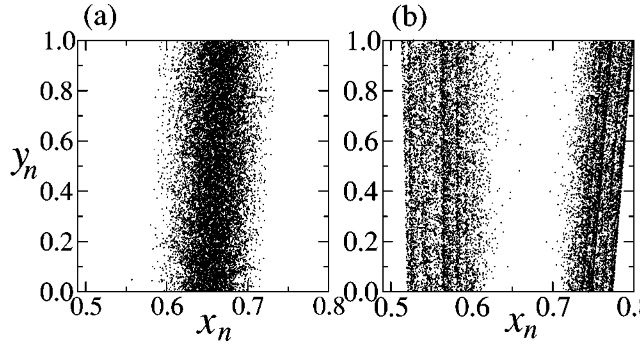

FIG. 6. Attractors before [(a) $\lambda = 2.95$] and after the bifurcation [(b) $\lambda = 3.10$] when $\epsilon = 0.2$.

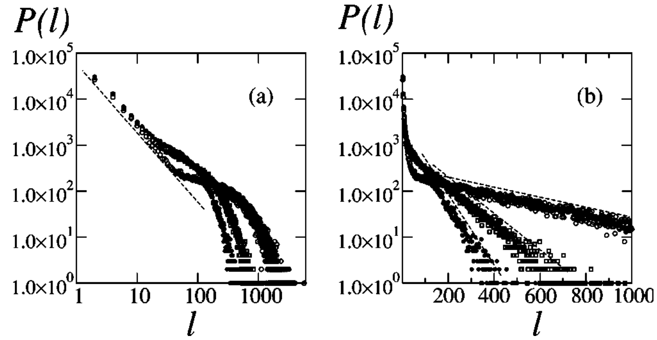

FIG. 7. Distribution function of the laminar length $P(l)$ near the onset of attractor bifurcation ($\lambda = 3.03$) for different noise levels ($\epsilon = 0.20$ (filled circle), $\epsilon = 0.15$ (square), and $\epsilon = 0.1$ (circle)] in (a) log-log and (b) linear-log scale.

As one can see in Fig. 2(b), there are two bands and each band corresponds to a new born attractor. Expanding Eq. (3) around the virtually invariant manifold, one sees that $x_{n+1} = \Lambda(y_n, \lambda, \epsilon) f(x^*) + \Lambda(y_n, \lambda, \epsilon) f'(x^*)(x_n - x^*) +$ higher order. Noting the definition of the virtually invariant manifold, one obtains the following equation:

$$\delta x_{n+1} = \eta_n \delta x_n + \epsilon \xi_n, \tag{7}$$

where $\delta x_n = x_n - x^*$, $\eta_n = \Lambda(y_n, \lambda, \epsilon) f'(x^*)$, and $\xi_n = (y_n - 0.5) f'(x^*)$. This is the normal form of on-off intermittency in the presence of additive noise [10]. Here we can see again that the coupling strength ϵ plays a role of noise level. For our system the driving term η_n has a bias due to the fact that actually our system is far from the invariant manifold. In other words, this bias is caused by introducing the virtually invariant manifold instead of a true invariant manifold. Accordingly, for given noise level, we can expect that there is a crossover time N^* for noise effects to become significant [10].

We can take one of two bands as a laminar phase and measure the laminar length l which is the duration time of signal below a threshold x_{th}. Figures 7(a) and 7(b) show the distribution function of the laminar length $P(l)$ in two different scales with $x_{th} = 0.69$. For short laminar length $l \ll N^*$, we can see the scaling rule obeys the well-known $-3/2$ power law in Fig. 7(a) which confirms that the dynamics is governed by on-off intermittency. And for long laminar length $l \gg N^*$ one clearly sees that $P(l)$ decays exponentially in Fig. 7(b). So in agreement with [10] our resulting scaling can be expressed as follows:

$$\begin{aligned} &P(l) \sim l^{-3/2} \quad \text{for } l < N^*(\epsilon), \\ &P(l) \sim \exp[-N^*(\epsilon) l] \quad \text{for } l > N^*(\epsilon). \end{aligned} \tag{8}$$

In Fig. 7(b), we can see that the slope of the exponential decay is getting smaller (slower mixing) as noise level is decreased which is also strong implication that dynamics is governed by on-off intermittency with noise [10]. Based on Eqs. (7) and (8) and numerical simulations in Fig. 7, we conclude that the unusual temporal behavior, which appears near the onset of attractor bifurcation, is the typical on-off intermittency phenomenon.

Equivalently, we can understand the exponential decay in our distribution function of the laminar length $P(l)$ in terms of crisis [6] and recurrence time of the attractor [11]. However, the $-3/2$ scaling which appears in short laminar length regime cannot be explained in this way. Accordingly, we conclude the observed phenomenon is better characterized by on-off intermittency than by crisis phenomenon, and it can be naturally understood in terms of on-off intermittency in the presence of noise. Moreover, based on the argument in [3,10] our result would not be changed for other chaotic or noise driving. Therefore, the described phenomenon is expected to be observed in many typical experimental situations.

In conclusion, we investigated a special bifurcation phenomenon, namely, attractor bifurcation, in a system of coupled chaotic maps and analyzed the underling mechanism of the phenomenon. Near the onset of bifurcation, the trajectory shows intermittency-like bursting behavior by alternately traveling around two new born attractors. We introduced the concept of the virtually invariant manifold and showed that our coupled system could be reduced to the normal form of on-off intermittency near the virtually invariant manifold in presence of noise, with which we explained the scaling rule in numerical simulations. All these observations lead us to the conclusion that the attractor bifurcation in our systems is governed by on-off intermittency in presence of noise. By this study we have elucidated the role of on-off intermittency and presented a coherent understanding of bifurcation phenomena in terms of on-off intermittency, which appear near the invariant manifold as well as far from it.

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